

Exact solutions of the 2D Schrödinger equation with central potentials induced by the non-commutativity of space

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Abstract

We obtain exact solutions of the 2D Schrödinger equation with the central potentials $V(r) = ar^2 + br^{-2} + cr^{-4}$ and $V(r) = ar^{-1} + br^{-2}$ in a non-commutative space up to the first order of noncommutativity parameter using the power-series expansion method similar to the 2D Schrödinger equation with the singular even-power and inverse-power potentials respectively in commutative space. We derive the exact non-commutative energy levels and show that the energy is shifted to m levels, as in the Zeeman effect.

KEYWORDS: non-commutative geometry, solutions of wave equations: bound states, algebraic methods.

PACS NUMBERS: 02.40.Gh, 03.65.Ge, 03.65.Fd

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1 Introduction

Non-commutative quantum mechanics is motivated by the natural extension of the usual quantum mechanical commutation relations between position and momentum, by imposing further commutation relations between position coordinates themselves. As in usual quantum mechanics, the non-commutativity of position coordinates immediately implies a set of uncertainty relations between position coordinates analogous to the Heisenberg uncertainty relations between position and momentum; namely:

$$[x^\mu, x^\nu]_* = i\theta^{\mu\nu}, \quad (1)$$

where $\theta^{\mu\nu}$ are the non-commutativity parameters of dimension of area that signify the smallest area in space that can be probed in principle. We use the symbol $*$ in equation (1) to denote the product of the non-commutative structure. This idea is similar to the physical meaning of the Plank constant in the relation $[x_i, p_j] = i\hbar\delta_{ij}$, which as is known is the smallest phase-space in quantum mechanics.

Our motivation is to study the effect of non-commutativity on the level of quantum mechanics when space non-commutativity is accounted for. One can study the physical consequences of this theory by making detailed analytical estimates for measurable physical quantities and compare the results with experimental data to find an upper bound on the θ parameter. The most obvious natural phenomena to use in hunting for non-commutative effects are simple quantum mechanics systems with central potential, such as the hydrogen atom [1–5]. In the non-commutative space one expects the degeneracy of the initial spectral line to be lifted, thus one may say that non-commutativity plays the role of magnetic field.

It has recently been shown that the non-inertial motion of the atom also induces corrections to the Lamb shift [6–9]. However, all the aforementioned studies are concerned with flat space-time. Therefore, it remains interesting to see what happens if the atom with central potential is placed in a non-commutative space rather than a flat one. In this work we present an important contribution to the non-commutative approach to the Schrödinger equation with central potentials. The study of the exact and approximate solutions of Schrödinger equation with central potentials has proved to be fruitful and many papers have been published [1 – 42]. Our goal is to solve the Schrödinger equation with singular even-power and inverse-power potentials induced by the non-commutativity of space. We thus find the exact non-commutative energy levels and that the non-commutativity effects are similar to the Zeeman splitting in commutative space.

This paper is organized as follows. In section 2, we derive the deformed 2D Schrödinger equation for a central potentials $V(r) = ar^2 + br^{-2} + cr^{-4}$ and $V(r) = ar^{-1} + br^{-2}$ in non-commutative space. We exactly solve the deformed Schrödinger equation in closed form [41] and obtain the exact non-commutative energy levels. Finally, section 3 is devoted to a discussion.

2 Non-commutative Schrödinger equation

In this section we study the exact solutions of the Schrödinger equation for the potentials $V(r) = ar^2 + br^{-2} + cr^{-4}$ and $V(r) = ar^{-1} + br^{-2}$ in the non-commutative space. The

non-commutative model specified by eq. (1) is defined by a star-product, where the normal product between two functions is replaced by the \star -product:

$$(\varphi \star \psi)(x) = \varphi(x) \exp\left(\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu\overrightarrow{\partial}_\nu\right)\psi(y)\Big|_{x=y}. \quad (2)$$

In a canonical non-commutative space-space type, the non-commutative quantum mechanics is described by the following equation:

$$H(p, x) \star \psi(x) = E\psi. \quad (3)$$

This equation reduces to the usual one described by [9, 16]:

$$H(\hat{p}, \hat{x})\psi(x) = E\psi, \quad (4)$$

where

$$\hat{x}_i = x_i - \frac{\theta_{ij}}{2}p_j, \quad \hat{p}_i = p_i. \quad (5)$$

2.1 The potential $V(r) = ar^2 + br^{-2} + cr^{-4}$

We can write the deformed potential $V(r) = ar^2 + br^{-2} + cr^{-4}$ in non-commutative space up to $\mathcal{O}(\Theta^2)$ as:

$$V(\hat{r}) = \frac{a}{2}\theta L_z + ar^2 + br^{-2} + \left(c + \frac{b}{4}\theta L_z\right)r^{-4} + \frac{c}{2}\theta L_z r^{-6}, \quad (6)$$

which is similar to the singular even-power potential which was studied in ref [41].

The Schrödinger equation in a 2D non-commutative space in the presence of the potential $V(\hat{r})$ can be cast into:

$$\left(-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + V(\hat{r})\right)\psi(\hat{r}) = E\psi(\hat{r}). \quad (7)$$

The solution to eq. (7) in polar coordinates $(\hat{r}, \hat{\varphi})$ takes the separable form [41]:

$$\psi(\hat{r}) = r^{-1/2}R_m(\hat{r})e^{im\varphi}. \quad (8)$$

Then eq. (7) reduces to the radial equation up to $\mathcal{O}(\Theta^2)$:

$$\frac{d^2R_m(\hat{r})}{dr^2} + \left[\tilde{E} + \tilde{V}(r) - \frac{m^2 - 1/4}{r^2}\right]R_m(\hat{r}) = 0, \quad (9)$$

where

$$\tilde{V}(r) = ar^2 + br^{-2} + \tilde{c}r^{-4} + \tilde{d}r^{-6}, \quad (10)$$

and

$$\tilde{E} = E - \frac{a}{2}\theta m, \quad \tilde{c} = c + \frac{b}{4}\theta m, \quad \text{and} \quad \tilde{d} = \frac{c}{2}\theta m.$$

Equation (9) is similar to the radial Schrödinger equation with singular even-power potential [41]. To solve the equation (7), we write the radial functions as [8, 14]:

$$R_m(\hat{r}) = e^{\tilde{p}_{|m|}(r)} \sum_{n=0} \tilde{a}_n r^{2n+\tilde{\nu}}, \quad (11)$$

where

$$\tilde{p}_{|m|}(r) = \frac{\alpha}{2} r^2 + \frac{\tilde{\beta}}{2} r^{-2}. \quad (12)$$

Substituting eq. (11) into eq. (9) and equating the coefficients of $r^{n+\nu}$ to zero, we obtain:

$$\tilde{A}_n \tilde{a}_n + \tilde{B}_{n+1} \tilde{a}_{n+1} + \tilde{C}_{n+2} \tilde{a}_{n+2} = 0, \quad (13)$$

where

$$\tilde{A}_n = \tilde{E} + \alpha(1 + 2\tilde{\nu} + 4n) \quad (14)$$

$$\tilde{B}_{n+1} = -b - 2\alpha\tilde{\beta} - \left(m^2 - \frac{1}{4}\right) + (\tilde{\nu} + 2n)(\tilde{\nu} - 1 + 2n) \quad (15)$$

$$\tilde{C}_n = \tilde{\beta}(3 - 2\tilde{\nu} - 4n) - \tilde{c}, \quad (16)$$

and

$$\alpha^2 = a, \tilde{\beta}^2 = \tilde{d}. \quad (17)$$

We can choose α and $\tilde{\beta}$ such that [41]:

$$\alpha = -\sqrt{a}, \tilde{\beta} = \sqrt{|\tilde{d}|}. \quad (18)$$

If $a_0 \neq 0$, then one obtains $C_0 = 0$, a condition that forbids the existence of the s energy levels ($|m| = 2l + 1$ in 2D). This is in fact a particularity of the non-commutative Schrödinger equation solution, which is not present in the ordinary Schrödinger framework [41]. Then we obtain:

$$\begin{aligned} \tilde{\nu} &= \left(\frac{3}{2} + \frac{\tilde{c}}{2\tilde{\beta}}\right) \\ &= \left(\frac{3}{2} + \tilde{\gamma}\right), \end{aligned} \quad (19)$$

where

$$\tilde{\gamma} = \frac{\tilde{c}}{2\tilde{\beta}}. \quad (20)$$

However if $a_n \neq 0$, with $a_{n+1} = a_{n+2} = \dots = 0$ then $\tilde{A}_n = 0$, from which one obtains the non-commutative energy eigenvalues exact up to $\mathcal{O}(\Theta^2)$:

$$\tilde{E}_{n,m} = \sqrt{a}(4 + 2\tilde{\gamma} + 4n) + \frac{a}{2}\theta m, \quad |m| = 1, 2, 3, \dots \quad (21)$$

We have thus shown that the degeneracy with respect to the angular quantum number m is removed and that non-commutativity here acts like a Lamb shift.

Now, we discuss the corresponding exact solution for $n = 1$. From eq. (21) the non-commutative energy splitting of the energy levels up to $\mathcal{O}(\Theta^2)$ is:

$$\begin{aligned}\tilde{E}_{1,m} &= \sqrt{a}(8 + \tilde{\gamma}) + \frac{a}{2}\theta m \\ &= \sqrt{a}\left(8 + \frac{c}{\sqrt{|\tilde{d}|}}\right) + \frac{a}{4}\left(2 + \frac{b}{\sqrt{a}}\right)\theta m \\ &= \sqrt{a}(8 + \tilde{\lambda}) + \frac{a}{4}(2 + \delta)\theta m,\end{aligned}\tag{22}$$

where

$$\tilde{\lambda} = \frac{c}{\sqrt{|\tilde{d}|}}, \quad \delta = \frac{b}{\sqrt{a}}\tag{23}$$

We have show that the non-commutative energy splitting is similar to the Zeeman effects and removes the degeneracy with respect to m . Furthermore we can say that the displacement of the energy levels is actually induced by the space non-commutativity which plays the role of a magnetic field. The corresponding eigenfunction is:

$$\psi_1(\hat{r}) = (\tilde{a}_0 + \tilde{a}_1 r^2) r^{\tilde{\nu}-1/2} e^{-\frac{1}{2}(\sqrt{a}r^2 + \sqrt{|\tilde{d}|}r^{-2})} e^{im\varphi},\tag{24}$$

where \tilde{a}_0 and \tilde{a}_1 can be calculated from eq. (13) and the normalisation condition. Following this method, we can obtain a class of exact solutions.

2.2 The potential $V(r) = ar^{-1} + br^{-2}$

The deformed potential $V(r) = ar^{-1} + br^{-2}$ in non-commutative space up to $\mathcal{O}(\Theta^2)$ is:

$$V(\hat{r}) = ar^{-1} + br^{-2} + \tilde{c}r^{-3} + \tilde{d}r^{-4},\tag{25}$$

where

$$\tilde{c} = \frac{a}{2}\theta m, \quad \text{and} \quad \tilde{d} = \frac{b}{4}\theta m,\tag{26}$$

where the third term is the dipôle-dipôle interaction created by the non-commutativity, the second term is a similar to the interaction between an ion and a neutral atom created by the non-commutativity. These interactions show us that the effect of space non-commutativity on the interaction of a single-electron atom, for example, is similar to that of a charged ion interacting with the atom on the one hand and on the other hand interacting with the electron to create a dipole and with the nucleus to create a second dipole.

The approach of the potential in eq. (25) is similar to that for the inverse-power potential in a commutative space. Thus we can take as solutions the eigenfunctions from Ref. [42]:

$$R_m(\hat{r}) = h_m(\hat{r}) e^{f(\hat{r})}, \quad m = 1, 2, 3, \dots\tag{27}$$

where

$$f(\hat{r}) = Ar^{-1} + Br + C \log r, \quad A \prec 0 \quad \text{and} \quad B \prec 0,\tag{28}$$

and

$$h_m(\hat{r}) = \prod_{j=1}^m (r - \tilde{\sigma}_j^m) = \sum_{j=1}^m \tilde{a}_j r^j. \quad (29)$$

Then the radial Schrödinger eq. (21) reduces to the following equation:

$$\left[f'' + f'^2 + \frac{h_m'' + 2h_m' f'}{h_m} + E - V(\hat{r}) - \frac{m^2 - 1/4}{r^2} \right] R_m(\hat{r}) = 0. \quad (30)$$

We arrive at the equation [42]:

$$f'' + f'^2 + \frac{h_m'' + 2h_m' f'}{h_m} = -E + V(\hat{r}) - \frac{m^2 - 1/4}{r^2}. \quad (31)$$

Now using the fact that:

$$f'' + f'^2 = B^2 + \frac{2BC}{r} - \frac{2AB}{r^2} + \frac{2A - 2AC}{r^3} + \frac{A^2}{r^4}, \quad (32)$$

and

$$h_m' = \sum_{j=1}^m j \tilde{a}_j r^{j-1}, \quad h_m'' = \sum_{j=1}^m j(j-1) \tilde{a}_j r^{j-2}, \quad (33)$$

where

$$\begin{aligned} a_m &= 1 \\ a_{m-1} &= -\sum_{j=1}^m \tilde{\sigma}_j^m \\ a_{m-2} &= -\sum_{j < i}^m \tilde{\sigma}_j^m \tilde{\sigma}_i^m \end{aligned} \quad (34)$$

and so on, then eqs. (31)-(33) lead to an algebraic equation where we equate equivalent coefficients of r^s between both sides of the equation, taking into account the eq. (34), we find:

$$A^2 = \tilde{d}, \quad \tilde{E} = -B^2 \quad (35)$$

$$2A(1 - C) = \tilde{c} \quad (36)$$

$$a = 2B(C + m), \quad (37)$$

and

$$\lambda = b + m^2 - 1/4 = C(C + 2m - 1) + m(m - 1) - 2B \left(A - \sum_{j=1}^m \sigma_j^m \right), \quad (38)$$

and

$$m\sqrt{\tilde{d}} + (m + 1 + C) \sum_{j=1}^m \sigma_j^m + B \sum_{j=1}^m (\sigma_j^m)^2 = 0, \quad (39)$$

$$(m-1) \sqrt{\tilde{d}} \sum_{j=1}^m \sigma_j^m + 2(m-1+C) \sum_{j \prec i} \sigma_j^m \sigma_i^m + B \sum_{j \prec i} \sigma_j^m \sigma_i^m \sum_{l \prec k} (\sigma_l^m + \sigma_k^m) = 0, \quad (40)$$

$$(m-2) \sqrt{\tilde{d}} \sum_{j \prec i} \sigma_j^m \sigma_i^m + 3(m-2+C) \sum_{j \prec i \prec k} \sigma_j^m \sigma_i^m \sigma_k^m + \\ + B \sum_{j \prec i \prec k} \sigma_j^m \sigma_i^m \sigma_k^m \sum_{l \prec q \prec s} (\sigma_l^m + \sigma_q^m + \sigma_s^m) = 0, \quad (41)$$

Moreover, multiplying equation (38) by B and using eqs.(35) - (37) we find the following algebraic equation for B as:

$$4 \left(A - \sum_{j=1}^m \sigma_j^m \right) B^2 + 2\tilde{\omega}B - a(\tilde{\nu} + 2m) = 0, \quad (42)$$

where

$$\tilde{\omega} = \lambda + m(2m + \tilde{\nu}) - m(m-1), \quad (43)$$

and

$$\tilde{\nu} = C - 1 = \frac{\tilde{c}}{2\sqrt{\tilde{d}}}. \quad (44)$$

The equation (42) is solved by:

$$B_{\pm} = \frac{\tilde{\omega} \pm \sqrt{\tilde{\omega}^2 + 4 \left(A - \sum_{j=1}^m \sigma_j^m \right) a(\tilde{\nu} + 2m)}}{4 \left(A - \sum_{j=1}^m \sigma_j^m \right)} \\ = \tilde{\omega} \frac{1 \pm \sqrt{1 + 4 \left(A - \sum_{j=1}^m \sigma_j^m \right) \frac{a(\tilde{\nu} + 2m)}{\tilde{\omega}^2}}}{4 \left(A - \sum_{j=1}^m \sigma_j^m \right)}. \quad (45)$$

So the non-commutative energy spectrum up to $\mathcal{O}(\Theta^2)$ is given by:

$$\tilde{E} = -\tilde{\omega}^2 \frac{\left(1 \pm \sqrt{1 + 4 \left(A - \sum_{j=1}^m \sigma_j^m \right) \frac{a(\tilde{\nu} + 2m)}{\tilde{\omega}^2}} \right)^2}{16 \left(A - \sum_{j=1}^m \sigma_j^m \right)^2}, \quad (46)$$

where

$$\tilde{\omega}^2 = \theta m \frac{a^2}{4b} \left(m^2 + \frac{2m}{\tilde{\nu}} (\lambda + m(m+1)) \right) + (\lambda + m(m+1))^2. \quad (47)$$

We have thus shown that the non-commutativity effects are manifested in energy levels, so that they are split into m levels, similarly to the effects of the magnetic field. Thus we can say that the non-commutativity plays the role of the magnetic field. It is also found that if the limit $\theta \rightarrow 0$ is taken, then we recover the results of the commutative case [42].

3 Conclusions

In this paper we started from a quantum particle with the central potentials $V(r) = ar^2 + br^{-2} + cr^{-4}$ and $V(r) = ar^{-1} + br^{-2}$ in a canonical non-commutative space. Using the Moyal product method, we have derived the deformed Schrödinger equation, we showed that it is similar to the Schrödinger equation with singular even-power and inverse-power potentials in commutative space. Using the power-series expansion method we solved it exactly and we found that the non-commutative energy is shifted to m levels. The non-commutativity acts here like a Lamb shift. This proves that the non-commutativity has an effect similar to the Zeeman effects, where the non-commutativity leads the role of the magnetic field.

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